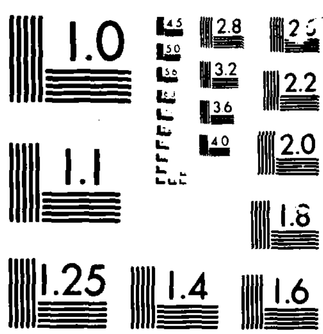
AD-A167 518 ON TRAVELLING-WAVE SOLUTIONS TO SYSTEMS OF CONSERVATION LAWS WITH SINGULA..(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER H PRUE ET AL. JAN 86 1/1

UNCLASSIFIED MRC-TSR-2906 DAAG29-80-C-0041 F/G 20/4 NL



MICROCOP CHART

2

AD-A167 518

MRC Technical Summary Report #2906

ON TRAVELLING-WAVE SOLUTIONS TO SYSTEMS OF CONSERVATION LAWS WITH SINGULAR VISCOSITY

Howard Prue and Richard Sanders

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

DTIC
S ELECTE D
MAY 2 3 1986
D

DTIC FILE COPY

January 1986

(Received October 30, 1985)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

86 5 20 147

86 5 20 147

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

ON TRAVELLING-WAVE SOLUTIONS TO SYSTEMS OF CONSERVATION LAWS WITH SINGULAR VISCOSITY

Howard Prue* and Richard Sanders**

Technical Summary Report #2906
January 1986

ABSTRACT

The method of vanishing artificial viscosity is used to obtain smooth, large-data travelling-wave solutions to a class of conservation laws with semidefinite viscosity. The one-dimensional Navier-Stokes equations serve as an illustrating example.

AMS (MOS) Subject Classifications: 35K65, 35L65.

Key Words: compressible Navier-Stokes equations, entropy inequality, artificial viscosity, modulo-2 intersection number

Work Unit Number 1 (Applied Analysis)

QUALITY INSPECTED 3

---

*CADAM Inc., 1935 N. Buena Vista, Burbank, California 91504.

**Department of Mathematics, University of Houston, Houston, Texas 77004.
This work was completed while the author was visiting the University of Wisconsin-Madison. Research supported by the National Science Foundation under Grant No. MCS 82-00676. Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

odes
, or
Dist
A-1

ON TRAVELLING-WAVE SOLUTIONS TO SYSTEMS OF CONSERVATION LAWS WITH SINGULAR VISCOSITY

Howard Prue* and Richard Sanders**

§1. INTRODUCTION

In this paper we discuss sufficient conditions for the existence of smooth, large-data travelling-wave solutions to certain systems of nonlinear, semidefinite parabolic partial differential equations. Specifically, we consider conservation laws which have the form:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \frac{\partial}{\partial x}\left(A(u)\,\frac{\partial u}{\partial x}\right) , \tag{1.1}$$

where u, $f(u) \in R^n$ and where $A(u) \in R^{n\times n}$ may be singular. Moreover, we include in this study problems for which $f(u)$ and $A(u)$ are not globally defined. To illustrate our ideas we apply the abstract results of this paper to the one-dimensional compressible Navier-Stokes equations; see [6] for a classical treatment of this problem. In the Navier-Stokes equations the left hand side of (1.1) is given by:

$$u = \begin{bmatrix} \rho \\ m \\ e \end{bmatrix} , \qquad f(u) = \begin{bmatrix} m \\ m^2/\rho + P \\ m(e + P)/\rho \end{bmatrix} ,$$

where ρ, m and e are respectively a fluids density, momentum and total energy. The variable P represents the fluids pressure and is given by:

$$P = (\gamma - 1)(e - m^2/2\rho) ,$$

for an ideal, polytropic gas. $\gamma > 1$ is a thermodynamic constant. The viscosity matrix $A(u)$ is given by:

$$\mu \begin{bmatrix} 0 & 0 & 0 \\ -\frac{m}{\rho^2} & \frac{1}{\rho} & 0 \\ -\left[\left(1 - \frac{\lambda}{c_v}\right)\frac{m^2}{\rho^3} + \frac{\lambda}{c_v}\frac{e}{\rho^2}\right] & \left[\left(1 - \frac{\lambda}{c_v}\right)\frac{m}{\rho^2}\right] & \left[\frac{\lambda}{c_v}\frac{1}{\rho}\right] \end{bmatrix} ,$$

*CADAM Inc., 1935 N. Buena Vista, Burbank, California 91504.

**Department of Mathematics, University of Houston, Houston, Texas 77004.
This work was completed while the author was visiting the University of Wisconsin-Madison. Research supported by the National Science Foundation under Grant No. MCS 82-00676. Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

where again λ and c_v are thermodynamic constants, and μ is a positive constant often referred to as the coefficient of viscosity.

A travelling-wave solution is a particular solution of (1.1) which has the form:

$$u(x,t) = w(x - st) , \tag{1.2}$$

$$\lim_{x \to \pm\infty} w(x - st) = u_{\pm} ,$$

for some scalar constant s. Inserting the Ansatz (1.2) into (1.1) and integrating we find that $w(\tau)$ satisfies the ordinary differential equations

$$-s(w - u_-) + f(w) - f(u_-) = A(w) \frac{dw}{d\tau} , \tag{1.3}$$

where here τ represents the travelling-wave variable $x - st$. (In later sections we use t to represent the travelling-wave variable.) For a trajectory of (1.3) to satisfy (1.2) it is easily seen that given u_- we must have that u_+ and s satisfy the Rankine-Hugoniot condition:

$$-s(u_+ - u_-) + f(u_+) - f(u_-) = 0 . \tag{1.4}$$

(From time to time we shall write s as $s(u_-,u_+)$ to indicate its explicit dependence on the states u_- and u_+.) Given that u_-, u_+ and s satisfy the Rankine-Hugoniot condition, it is a natural question to ask whether there is a smooth function $w(\tau)$ that satisfies (1.2), (1.3). To answer this question when the viscosity matrix $A(u)$ is singular our approach is to perturb $A(u)$ to a nonsingular matrix $A^{\varepsilon}(u)$ (with $A^0(u) = A(u)$) and establish the existence of a family $\{w^{\varepsilon}\}$ of solutions to the modified problems. The behavior of w^{ε} as ε tends to zero is then investigated. For the Navier-Stokes equations the modified problems are obtained by introducing the artificial viscosity

$$\varepsilon \frac{\partial}{\partial x} \left[a(u) \frac{\partial}{\partial x} \left(\log\left(\frac{\rho^{\gamma}}{e - m^2/2\rho}\right) - \frac{m^2}{2\rho^2(e - m^2/2\rho)}\right)\right] , \tag{1.5}$$

(with $a(u) > 0$ to be determined), into the conservation of mass equation.

This paper is divided into five sections. In Section 2 we list a number of preliminary assumptions. Section 3 is devoted to the existence of travelling-wave solutions to (1.1) when the viscosity matrix $A(u)$ is nonsingular. Singular viscosity problems are treated in Section 4. Finally, in Section 5 we apply our results to the compressible Navier-Stokes equations.

Acknowledgement. We wish to thank Professor J. V. Ralston of UCLA in particular for suggesting the proof of Theorem 1 as well as for his other invaluable comments.

§2. SOME PRELIMINARY ASSUMPTIONS

The global structure of the Hugoniot locus (that is the set of states u_+ that satisfy (1.4) for some fixed u_- and variable $s \in \mathbb{R}$) has fundamental importance to our analysis. To begin let R denote some open connected subset of $\mathbb{R}^n$. The set R, which is usually determined from the physics of (1.1), should be regarded as the set of physically admissible states. We first require that

(2.1) $$f(u) \in C^3(R) ,$$

and we assume that the eigenvalues of $Df(u)$ are real, distinct and are arranged in increasing order:

(2.2) $$\lambda_1(u) < \lambda_2(u) < \dots < \lambda_n(u) .$$

In addition to this, we require that for each $\lambda_k(u)$ and all $u \in R$ either

(GNL) $$\nabla_u \lambda_k(u) \cdot r_k(u) \neq 0 ,$$

or

(LD) $$\nabla_u \lambda_k(u) \cdot r_k(u) \equiv 0 ,$$

is satisfied, where $r_k(u)$ is the right eigenvector of $Df(u)$ corresponding

to $\lambda_k(u)$; see [8]. Fields that satisfy (GNL) are often referred to as "genuinely nonlinear" fields, and fields that satisfy (LD) are referred to as "linearly degenerate".

For $u_- \in R$ it is well known that (2.2) implies that the Hugoniot locus is locally the union of n connected one-dimensional manifolds. Following Mock [11] we denote the Hugoniot locus about u_- as

$$\Gamma(u_-) = \bigcup_{k=1}^{n} \Gamma_k(u_-) , \tag{2.3}$$

where the index k is chosen such that the tangent vector of $\Gamma_k(u_-)$ at u_- is $r_k(u_-)$. We require that the local structure of $\Gamma(u_-)$ is valid in the large and that R is chosen so that for any $u_- \in R$ we have

$$\Gamma(u_-) \subset R . \tag{2.4}$$

Next we impose a global "entropy condition" on those fields which satisfy (GNL), see [8]. Specifically, if $u_+ \in \Gamma_k(u_-)$ where $\lambda_k(u)$ satisfies (GNL), and when $\lambda_k(u_-) > \lambda_k(u_+)$, we require that

$$\begin{aligned} &\lambda_k(u_-) > s(u_-,u_+) > \lambda_k(u_+) , \\ &\lambda_{k+1}(u_+) > s(u_-,u_+) > \lambda_{k-1}(u_-) , \end{aligned} \tag{2.5}$$

where $s(u_-,u_+)$ is the "shock speed" determined by the Rankine-Hugoniot condition. (2.5) is of course the celebrated Lax condition E.

Remark 2.1: For those $\lambda_k(u)$'s that satisfy (GNL) one easily deduces that (2.5) partitions $\Gamma_k(u_-)$ into two disjoint branches separated by u_-. On one branch (2.5) is satisfied. (Throughout we shall denote this branch by $\Gamma_k^-(u_-)$.) On the other branch (2.5) is satisfied with the roles of u_- and u_+ reversed.

Finally, we assume that $f(u)$ admits a convex entropy function. A convex entropy function, say $E(u)$, is a scalar-valued function which

satisfies:

(2.6a) $E(u) \in C^3(\mathcal{R})$,

(2.6b) $D^2E(u)$ is positive definite on $\mathcal{R}$,

(2.6c) $D^2E(u)Df(u)$ is symmetric.

Given that $n = 2$ and (2.2) is satisfied, such a function often exists [9]. Frequently this is not always the case when $n \geq 3$. Since, as will be seen below, the existence of $E(u)$ is crucial for our analysis, we should mention that numerous physical problems do indeed admit a convex entropy function [4].

We now state two lemmas without proof. Their proofs can easily be obtained by adapting Lemma 5 and Theorem 3 from Mock [11].

<u>Lemma 2.1</u>. Suppose that all of the assumptions above (with the possible exception of (2.6)) are satisfied. Let $u_+ \in \Gamma_k(u_-)$, $u_+ \neq u_-$, where $\lambda_k(u)$ satisfies (GNL). Then the function

$$H(u_-,u_+;u) = -s(u_-,u_+)(u - u_-) + (f(u) - f(u_-)) ,$$

has no zeros in $\mathcal{R}$ other than u_- and u_+. Moreover, given u_- and u_+ and any open set $\mathcal{O}$ containing u_- and u_+, there is a positive constant $c(u_-,u_+,\mathcal{O})$ such that for any $u \in \mathcal{R}\backslash\mathcal{O}$

$$|H(u_-,u_+;u)| \geq c(u_-,u_+,\mathcal{O}) .$$

<u>Lemma 2.2</u>. Given that $E(u)$ satisfies (2.6), there exists a smooth scalar-valued function $F(u)$ such that

$$(Df(u))^T \nabla E(u) = \nabla F(u) .$$

Moreover, if u_+ ($\neq u_-$) lies in a genuinely nonlinear field on which the entropy condition (2.5) is satisfied (i.e. $u_+ \in \Gamma_k^-(u_-)$), then

$$s(u_-,u_+)(E(u_+) - E(u_-)) - (F(u_+) - F(u_-)) > 0 .$$

On the other hand, if u_+ lies in a linearly degenerate field the inequality above becomes an equality.

We conclude this section by showing that the flux term $f(u)$ from the Navier-Stokes equations satisfies all of the assumptions so far imposed. Recall from Section 1 that the flux terms of the Navier-Stokes equations are given by:

$$f(u) = \begin{bmatrix} m \\ m^2/\rho + P \\ m(e + P)/\rho \end{bmatrix} ,$$

where ρ is the density of the fluid, m is the momentum and e is the total energy (kinetic plus internal energy). The pressure term above is given by:

$$P = (\gamma - 1)\left(e - \frac{1}{2} m^2/\rho\right) ,$$

for an ideal, calorically perfect gas, where $\gamma > 1$ is a thermodynamic constant. The eigenvalues of $Df(u)$ are:

$$\lambda_1(u) = m/\rho - C, \quad \lambda_2(u) = m/\rho, \quad \lambda_3(u) = m/\rho + C , \tag{2.7}$$

where C, the sound speed, is given by:

$$C^2 = \gamma P/\rho .$$

Furthermore, $\lambda_1(u)$ and $\lambda_3(u)$ satisfies (GNL), while $\lambda_2(u)$ satisfies (LD). The Hugoniot locus for this problem is readily computed to be [12]:

$$\Gamma_1(u_-) = \begin{cases} P_+ = P_-\theta^{-1} \\ \rho_+ = \rho_-(\beta + \theta)/(1 + \beta\theta) \\ (m/\rho)_+ = (m/\rho)_- + C_- \dfrac{2\sqrt{\tau}}{\gamma - 1} \cdot \dfrac{1 - \theta^{-1}}{(1 + \beta\theta^{-1})^{1/2}} , \end{cases} \tag{2.8}$$

$$\Gamma_2(u_-) = \begin{cases} \rho_+ = P_- \\ \rho_+ = \rho_-\theta \\ (m/\rho)_+ = (m/\rho)_- , \end{cases}$$

$$\Gamma_3(u_-) = \begin{cases} P_+ = P_-\theta \\ \rho_+ = \rho_-(1 + \beta\theta)/(\beta + \theta) \\ (m/\rho)_+ = (m/\rho)_- + C_- \dfrac{2\sqrt{\tau}}{\gamma - 1} \cdot \dfrac{\theta - 1}{(1 + \beta\theta)^{1/2}} , \end{cases}$$

where $\beta = \frac{\gamma + 1}{\gamma - 1}$ and $\tau = \frac{\gamma - 1}{2\gamma}$ are constants and the parameter θ lies in $\theta > 0$. Moreover, in the 1 and 3 fields above the entropy condition (2.5) is satisfied when $0 < \theta < 1$; see Smoller's book [12] for a thorough development of these facts.

We take for the set of physically admissible states the convex set

$$R = \{(\rho,m,e)^T : \rho > 0,\ P > 0\} ,$$

and one easily finds that $\Gamma(u_-) \subset R$ for any $u_- \in R$. It is furthermore well known that the ideal, calorically perfect Navier-Stokes equations admit a convex entropy function

$$E(u) = -\rho \log (P/((\gamma-1)\rho^\gamma)) \quad ,$$

see [4]. Clearly $E(u) \in C^3(R)$ and a lengthy calculation will verify that $E(u)$ above also satisfies (2.6b) and (2.6c) [4, 1].

We therefore have that the flux terms of the Navier-Stokes equations fit completely into the framework we have so far developed. We should mention however that not all of the hypotheses above are sacred (other than (2.6) that is). We have chosen the route of simplicity over generality here since such an interesting physical example is so easily accommodated.

§3. LARGE-DATA TRAVELLING-WAVE SOLUTIONS TO POSITIVE DEFINITE SYSTEMS

In this section we state sufficient hypotheses and outline the technique we use to establish the existence of large-data travelling-wave solutions to a particular version of (1.1). The specific simplifying assumption we make here, in addition to the assumptions of Section 2, is that the "effective"

diffusion matrix, D^2EA, has positive definite symmetric part. Although the main result of this section is a generalization of previously known results [5, 2, 3, 11], the positivity assumption is unreasonable when considering problems such as the compressible Navier-Stokes equations or other physical examples with singular viscosity matrices. Nevertheless our approach motivates a systematic procedure for constructing reasonable artificial viscosity terms for which uniform estimates are obtained. With these estimates the vanishing artificial viscosity method can be applied.

Remark 3.1. The condition that D^2EA have positive definite symmetric part is equivalent to the condition that D^2EA^{-1} have positive definite symmetric part.

We use the method of continuation to obtain a large-data travelling-wave solution. A homotopy invariant is constructed which if equal to unity guarantees the existence of a travelling-wave. The problem is then smoothly deformed to a small-data (weak-wave) problem for which it is routinely seen (via an argument similar to Foy [5]) that the invariant is in fact unity. Before stating the main result of this section (Theorem 1) we outline our method of proof.

Let $\Gamma_k^-(u_-)$ denote the branch of the Hugoniot locus on which the entropy condition (2.5) is satisfied, (of course we assume that $\lambda_k(u)$ satisfies (GNL)). We suppose that u_+ lies on this branch and that $u_+(\sigma)$ is the arc-length parameterization along $\Gamma_k^-(u_-)$ with $u_+(0) = u_-$ and $u_+(L) = u_+$. Now consider the variable wave strength version of (1.1) with its associated travelling-wave equation:

(3.1) $$A(w)\frac{dw}{dt} = H(u_-,u_+(\sigma);w) ,$$

$$\lim_{t\to-\infty} w(t) = u_-, \qquad \lim_{t\to+\infty} w(t) = u_+(\sigma) ,$$

where $H(u_-,u_+(\sigma);u)$ is given by:

$$-s(u_-,u_+(\sigma))(u - u_-) + f(u) - f(u_-) .$$

The scalar function

(3.2) $$\Lambda(u) = \nabla E(u) \cdot H(u_-,u_+(\sigma);u) + s(u_-,u_+(\sigma))(E(u) - E(u_-)) - (F(u) - F(u_-)) ,$$

will serve as a Lyapunov function. The fact that $\Lambda(u)$ is indeed a Lyapunov function is evident by differentiating (3.2) along trajectories of (3.1), giving

$$\frac{d}{dt}\Lambda(w) = D^2E(w)H(u_-,u_+(\sigma)w) \cdot A^{-1}(w)H(u_-,u_+(\sigma);w) .$$

(Recall from Lemma 2.2 that $Df^T\nabla E = \nabla F$.) Since by assumption D^2EA has positive definite symmetric part, Lemma 2.1 implies that $\frac{d}{dt}\Lambda(w(t)) > 0$ for all $w(t) \in \mathcal{R}$ except for u_- and $u_+(\sigma)$. Next we reparameterize t in (3.1) so that

(3.3) $$\frac{d}{dt}\Lambda(w(t)) = 1 .$$

Away from u_- and $u_+(\sigma)$ this is equivalent to multiplying the left hand side of (3.1) by the positive function

$$\mu(w) = D^2E(w)H(u_-,u_+(\sigma);w) \cdot A^{-1}(w)H(u_-,u_+(\sigma);w)$$

We wish to show that the unstable manifold of (3.1) near u_- can be connected to the stable manifold near u_+. With this in mind, consider the region in state-space given by:

(3.4) $$M_- = L_-(\varepsilon) \cap \mathcal{U}_- ,$$

where

$$L_-(\varepsilon) = \{u \in \mathcal{R} : \Lambda(u) = \varepsilon\}, \quad (\varepsilon > 0) ,$$

$$\mathcal{U}_- = \text{unstable manifold of (3.1) near } u_- .$$

We now give:

Claim 1. For all $0 < \sigma < L$, M_- is homeomorphic to the sphere S^{n-k} in a small neighborhood of u_- provided that $\varepsilon > 0$ is chosen sufficiently small. (Note that throughout this section k represents the index of the entropy condition satisfying branch of the Hugoniot curve on which $u_+(\sigma)$ lies.)

Before proving the claim we state a lemma from Mock [11].

Lemma 3.1. Let B and C be matrices and suppose that B has positive definite real part, $\mathrm{Re}(B) = 1/2\,(B + B^*)$, and that C is nonsingular and Hermitian. Then the form $q(z) = z^*Cz$ is positive definite on the generalized eigenspace of $B^{-1}C$ corresponding to those eigenvalues with positive real parts.

Proof of Claim 1: The proof follows from the assumptions of the previous section and Lemma 3.1. For $\varepsilon > 0$ sufficiently small, $L_-(\varepsilon)$ is a manifold of codimension 1. This is clear since the only critical values of $\Lambda(u)$ are $\Lambda(u_-) = 0$ and $\Lambda(u_+(\sigma)) > 0$ (see Lemma 2.2) and ε can be taken between these two values. The unstable manifold of (3.1) is given locally by the generalized eigenvectors of $A^{-1}(u_-)DH(u_-,u_+(\sigma);u_-)$ which correspond to those eigenvalues with positive real parts. Combining the facts that $D^2E(u_-)$ symmetrizes $DH(u_-,u_+(\sigma);u_-)$ and D^2EA has positive definite symmetric part, it is easily shown that the number of eigenvalues of $A^{-1}(u_-)DH(u_-,u_+(\sigma);u_-)$ with positive real parts is equal to the number of positive eigenvalues of $DH(u_-,u_+(\sigma);u_-)$. Counting this number (that is, using the entropy condition (2.5)) we conclude that $\dim(\mathcal{U}_-) = n - k + 1$, and the count is independent of $0 < \sigma < L$. Finally, using Lemma 3.1 with $C = D^2E(u_-)DH(u_-,u_+(\sigma);u_-)$ and $B = D^2E(u_-)A(u_-)$, we have that

$$D^2\Lambda(u_-) = D^2E(u_-)DH(u_-,u_+(\sigma);u_-) \ ,$$

is positive definite when acting on the tangent space of $\mathcal{U}_-$ near u_-. Thus, M_- is locally homeomorphic to the sphere S^{n-k}.

What follows below are the key points of this section. Let $\varphi_t(x)$ represent the flow of the system

$$\mu(w)A(w)\,\frac{dw}{dt} = H(u_-,u_+(\sigma);w)\ , \tag{3.5}$$

$$w(0) = x\ ,$$

and notice that φ_{t_0} (when defined) maps M_- into the codimension 1 manifold:

$$L_+(\varepsilon) = \{u \in \mathcal{R} : \Lambda(u) = \Lambda(u_+(\sigma)) - \varepsilon\}\ , \tag{3.6}$$

when $t_0 = \Lambda(u_+(\sigma)) - 2\varepsilon$ and where again we have taken $\varepsilon > 0$ small. Next define

$$M_+ = L_+(\varepsilon) \cap \mathcal{S}_+\ , \tag{3.7}$$

where

$$\mathcal{S}_+ = \text{stable manifold of (3.1) near } u_+(\sigma)\ .$$

We now give another simple claim:

<u>Claim 2</u>. For all $0 < \sigma \leq L$, M_+ is homeomorphic to the sphere S^{k-1} in a neighborhood of $u_+(\sigma)$ where as before $\varepsilon > 0$ is taken sufficiently small.

The homotopy invariant alluded to above is the modulo-2 intersection number of $\varphi_{t_0}(M_-)$ and M_+ defined inside the ambient space $L_+(\varepsilon)$; see [7]. As per Claim 1 and Claim 2, $\varphi_{t_0}(M_-)$ and M_+ have complimentary dimension, that is $\dim(\varphi_{t_0}(M_-)) + \dim(M_+) = \dim(L_+(\varepsilon))$, <u>provided</u> that $\varphi_{t_0} : M_- \to L_+(\varepsilon)$ is a smooth map. Unfortunately, for general large-data problems, this need not be the case. The next lemma addresses this matter in the small.

<u>Lemma 3.2</u>. The flow $\varphi_t(M_-)$, $0 \leq t \leq t_0$, remains in a compact subset of $\mathcal{R}$ for all $0 < \sigma \leq \sigma_0$, provided that σ_0 is taken sufficiently small.

Proof of Lemma 3.2: Observe that

$$\Lambda(u_+(\sigma)) = s(u_-,u_+(\sigma))(E(u_+(\sigma)) - E(u_-)) - (F(u_+(\sigma)) - F(u_-)) .$$

For weak and entropy satisfying shocks (i.e. $u_+(\sigma) \in \Gamma_k^-(u_-)$ with σ small) Lax has shown that

$$\Lambda(u_+(\sigma)) = O(\sigma^3) > 0 .$$

Computing $\frac{d}{ds}\Lambda(w(t(s)))$, where s is the arc length parameterization along a trajectory of (3.1), we obtain

$$\frac{d}{ds}\Lambda(w) = \frac{D^2E(w)H(u_-,u_+(\sigma);w) \cdot A^{-1}(w)H(u_-,u_+(\sigma);w)}{|A^{-1}(w)H(u_-,u_+(\sigma);w)|} ,$$

and it is easily seen that for any $w \in B(u_-,\delta)$ with $\delta > 0$ small, we have

$$\frac{d}{ds}\Lambda(w) > \text{const}(1 - \delta)|H(u_-,u_+(\sigma);w)| . \tag{3.8}$$

The positive constant above does not depend on δ. Expanding Mock's proof of Lemma 2.1 one determines that

$$|H(u_-,u_+(\sigma);w)| > \text{const}\ \delta\sigma , \tag{3.9}$$

for $w \in R$ but outside $B(u_-,\delta) \cup B(u_+(\sigma),\delta)$, again for $\delta > 0$ and $\sigma > 0$ sufficiently small. We next show that $w(t)$ cannot leave $B(u_-,\rho) \cup B(u_+(\sigma),\rho)$ for an appropriate choice of ρ.

For any $0 < \underset{\sim}{t} < t < t_0$, we have from (3.3) that

$$\Lambda(u_+(\sigma)) > \int_{\underset{\sim}{t}}^{t} \frac{d}{dt}\Lambda(w)dt , \tag{3.10}$$

and we choose $\underset{\sim}{t}$ to be the first time that $w(t)$ $B(u_-,\delta) \cup B(u_+(\sigma),\delta)$. To reach a contradiction suppose that $w(t)$ leaves $B(u_-,\delta + \sigma) \cup B(u_+(\sigma),\delta + \sigma)$. Reparameterizing the right hand side of (3.10) in terms of arc length and using (3.8) and (3.9) we must have that

$$\Lambda(u_+(\sigma)) > \text{const}(1 - \delta - \sigma)\delta\sigma \cdot \sigma .$$

However, recalling that $\Lambda(u_+(\sigma)) = 0(\sigma^3)$, the inequality above would imply that

$$0(\sigma) > \text{const}(1 - \delta - \sigma)\delta .$$

Setting $\delta = \kappa\sigma$ with $\kappa < 1/(2\sigma)$, we reach an obvious contradiction for small σ and large κ. This completes the proof since we could take σ smaller if necessary to force $B(u_-,(\kappa + 1)\sigma) \cup B(u_+(\sigma),(\kappa + 1)\sigma)$ to be contained in R.

Remark 3.2. With the special assumption that $R = \mathbb{R}^n$ along with the assumptions that $D^2E(u)A^{-1}(u)$ has uniformly positive definite symmetric part and $A^{-1}(u)$ is uniformly bounded, one can show that the conclusion of the previous lemma remains valid independent of the size of σ.

The proof of the next proposition follows the proof of Theorem 1.

Proposition 1. Suppose that the preliminary assumptions of Section 2 are satisfied. Furthermore, assume that D^2EA has positive definite symmetric part in R, $u_- \in R$ and $u_+(\sigma) \in \Gamma_k^-(u_-)$, the entropy condition satisfying branch of $\Gamma_k(u_-)$. Then $I_2(\varphi_{t_0}(M_-),M_+)$, the modulo-2 intersection number, equals one provided that $0 < \sigma \leq \sigma_0$ with σ_0 sufficiently small.

Having additional knowledge concerning $\varphi_t(M_-)$ allows us to state:

Theorem 1. Suppose that the assumptions of Proposition 1 are satisfied along with the key hypothesis:

(A) $\varphi_t(M_-)$ for $0 \leq t \leq t_0$ remains in a compact subset of R for all $0 < \sigma \leq L$.

Then (1.1) admits a smooth travelling-wave solution for any $u_+(\sigma)$ with $0 < \sigma \leq L$.

Proof of Theorem 1: By hypothesis A, Claim 1 and Claim 2, together with the usual theorems from ordinary differential equations, we have that $\varphi_{t_0}(M_-)$ and M_+ are compact submanifolds of $L_+(\varepsilon)$. Moreover, φ_{t_0} and M_+ are

smoothly dependent on $0 < \sigma < L$, and they have complementary dimensions with respect to $L_+(\varepsilon)$. Therefore, the modulo-2 intersection number, $I_2(\varphi_{t_0}(M_-),M_+)$, is well defined and remains constant throughout the entire σ deformation. Proposition 1 establishes the fact that $I_2(\varphi_{t_0}(M_-),M_+) = 1$ for small σ_0, thus $I_2(\varphi_{t_0}(M_-),M_+) = 1$ for any $\sigma_0 < \sigma < L$. So we conclude that there exists at least one trajectory of (3.1), (with $\sigma = L$), that connects the unstable manifold near u_- to the stable manifold near $u_+(L)$. This is the desired result.

Proof of Proposition 1: First observe that (3.5) can be written as:

$$\mu(w)D^2E(w)A(w)\frac{dw}{dt} = \nabla\Lambda(w) \quad (3.11)$$

Applying the results of this section it is clear that we can deform $D^2E(w)A(w)$ to the identity without changing the modulo-2 intersection number, provided $0 < \sigma < \sigma_0$ and σ_0 is sufficiently small. Now let $\{r_\ell\}_{\ell=1}^n$ represent the orthonormal set of eigenvectors to the symmetric matrix

$$D^2E(u_-)DH(u_-,u_+(\sigma);u_-) = D^2\Lambda(u_-) , \quad (3.12)$$

arranged so that its eigenvalues $\tilde{\lambda}_\ell$ are increasing with ℓ. Consider the orthogonal change of coordinates $w - u_- = \sum_{\ell=1}^{n} v_\ell r_\ell$ and the scalar function

$$\Lambda^c(v) = \frac{1}{2}\sum_{\ell=1}^{n}\tilde{\lambda}_\ell(v_\ell)^2 + \frac{b}{6}(v_k)^3 ,$$

where b is given by

$$\sum_{\alpha,\beta,\gamma}(r_k)_\alpha(r_k)_\beta(r_k)_\gamma\frac{\partial^3\Lambda(u_-)}{\partial u_\alpha\partial u_\beta\partial u_\gamma} .$$

$\Lambda^c(v)$ contains the important local information of $\Lambda(w)$. A straightforward calculation shows that the eigenvalues of (3.12) are given by:

$$\tilde{\lambda}_\ell = \begin{cases} -(\beta_\ell)^2 + O(\sigma) & \text{for } \ell < k \\ \frac{1}{2}(\beta_k)^2\sigma + O(\sigma^2) & \text{for } \ell = k \\ +(\beta_\ell)^2 + O(\sigma) & \text{for } \ell > k , \end{cases}$$

and b can be rewritten as

$$b = -(\beta_k)^2 + O(\sigma) \ ,$$

where

$$(\beta_k)^2 = |\nabla\lambda_k(u_-) \cdot r_k|(r_k^T D^2 E(u_-) r_k) \ .$$

(Again, recall that the index k refers to the fact that $u_+(\sigma) \in \Gamma_k^-(u_-)$.) Defining $v' = (v_1, \ldots, v_{k-1}, 0, v_{k+1}, \ldots, v_n)^T$, we have from Taylor's Theorem that

$$\frac{\partial}{\partial v_k} (\Lambda(w) - \Lambda^c(v)) = O(|v_k||v'| + |v'|^2 + |v|^3) \ ,$$

and

$$\frac{\partial}{\partial v_\ell} (\Lambda(w) - \Lambda^c(v)) = O(|v|^2) \quad \text{for} \quad \ell \neq k \ .$$

Rotate coordinates, $(w - u_- = Rv)$, and consider the final deformation:

$$\mu(\eta;v) \frac{dv}{dt} = \nabla_v \Lambda(\eta;v) \ ,$$

where $\Lambda(\eta;v) = \Lambda^c(v) + \eta(\Lambda(w) - \Lambda^c(v))$. Following the outline previously laid down in this section (Claim 1, Claim 2 and Lemma 3.2) along with the estimates above it is not difficult to show that $I_2(\varphi_{t_0}(M_-), M_+)$ is well defined and remains constant for all $\eta \in [0,1]$ provided that $\sigma > 0$ is fixed and is sufficiently small. Setting $\eta = 0$ one easily verifies that $\varphi_{t_0}(M_-)$ intersects M_+ exactly once and that this intersection is indeed transversal. Therefore, the modulo-2 intersection number is 1 for $\eta = 0$ from which by homotopy invariance we conclude the same for $\eta = 1$.

Remark 3.3. We have intentionally excluded the case when u_+ lies in a linear degenerate field. In this case (3.1) can have a smooth solution only for trivial data $u_+(\sigma) = u_-$. This is seen by first recalling that

$$D^2E(u)H(u_-, u_+(\sigma); u) = \nabla\Lambda(u) \ ,$$

and then multiplying (3.1) by $\frac{dw}{dt}^T D^2E(w)$; doing so we obtain:

$$0 < \frac{dw}{dt}^T D^2E(w)A(w)\frac{dw}{dt} = \frac{d}{dt}\Lambda(w) .$$

Lemma 2.2 states that when $u_+(\sigma)$ is in a linear degenerate field $\Lambda(u_+(\sigma)) = \Lambda(u_-) = 0$, therefore the equation above allows us to draw only one conclusion, that is $w(t) = \text{const}$.

We end this section by giving sufficient conditions under which hypothesis A of Theorem 1 is satisfied. We state these conditions here as a theorem since it is this technique we apply in the following sections.

Theorem 2. Let $G \subset \mathbf{R}^n$ represent the set of states that can be reached by any smooth path $x(t)$ with $x(0) \in M_-$, $\frac{d}{dt}\Lambda(x(t)) > 0$ and $\varepsilon < \Lambda(x(t)) < \Lambda(u_+) - \varepsilon$. Suppose first that

(a) $$\bar{G} \subset R .$$

Second, assume that there exists a smooth, nondecreasing function $g(r)$, with $\lim_{r\to\infty} g(r) = +\infty$, which satisfies

(b) $$\min_{|\xi|=1} [\xi^T\mu(u)D^2E(u)A(u)\xi]^{1/2} > g'(|u|) ,$$

for all $u \in \bar{G}$. Then $\varphi_t(M_-)$ remains in a compact subset of R for any $0 < t < \Lambda(u_+) - 2\varepsilon$.

Proof: Clearly by assumption (a) we have that $\varphi_t(M_-)$ remains in a closed subset of R. To see that $\varphi_t(M_-)$ remains bounded, let $m_- \in M_-$ and observe that

$$[g(|w(t)|) - g(|m_-|)]^2 = [\int_0^t \frac{d}{d\tau} g(|w(\tau)|)d\tau]^2 < t\int_0^t |\frac{d}{d\tau} g(|w(\tau)|)|^2 d\tau .$$

Computing the derivative in the right hand side above and using assumption (b) we arrive at

$$[g(|w(t)|) - g(|m_-|)]^2 < t\int_0^t \frac{dw}{d\tau}^T \mu(w)D^2E(w)A(w)\frac{dw}{d\tau}d\tau .$$

Using (3.11) we find that the integrand above is equal to $\frac{d}{d\tau}\Lambda(w)$, and recall from (3.3) that this quantity is normalized to 1. Therefore we conclude that

$$g(|w(t)|) < g(|m_-|) + t < g(|m_-|) + \Lambda(u_+) - 2\epsilon ,$$

which implies the desired result.

§4. SINGULAR VISCOSITY MATRICES

We carry over all of the hypotheses of Section 2 to investigate problem (1.1) in the case when the viscosity matrix A is singular. We furthermore assume the set R of physically admissible states is convex. The convexity of R along with the assumption that $D^2E(u)$ is positive definite on R allows us to introduce the globally defined change of coordinates:

$$v = \nabla E(u) . \tag{4.1}$$

In the new coordinates, the travelling-wave differential equation (3.1) becomes:

$$B(v)\frac{dv}{dt} = H(u_-,u_+;u(v)) , \tag{4.2}$$

$$\lim_{t\to\pm\infty} v(t) = v_\pm \equiv \nabla E(u_\pm) ,$$

where $B(v) = A(u(v))(D^2E(u(v)))^{-1}$. Throughout, $B(v)$ is assumed to have positive semi-definite symmetric part. In the coordinate system given by (4.1), the right hand side of (4.2) is given by:

$$H(u_-,u_+;u(v)) = \nabla_v\Lambda(u(v)) ,$$

(recall that $\Lambda(u)$ is defined in equation (3.2)). To simplify the notation below we define $\bar{\Lambda}(v)$ by:

$$\bar{\Lambda}(v) \equiv \Lambda(u(v)) .$$

<u>Remark 4.1</u>. We choose here to work in the coordinate system (4.1) because it is this coordinate system we use in the application of the next section.

We now make some further simplifying assumptions. Suppose that the null-space of the matrix $B(v)$ is spanned by p independent constant vectors. This allows us to make a simple rotation (which below we take to be the identify) so that $B(v)$ may be partitioned as:

$$B(v) = \begin{bmatrix} 0 & 0 \\ \underset{\sim}{b}(v) & b(v) \end{bmatrix}, \tag{4.4}$$

where $b(v)$ is a $(n - p) \times (n - p)$ positive matrix and $\underset{\sim}{b}(v)$ is an $(n - p) \times p$ matrix. To further simplify our presentation we assume that $\underset{\sim}{b}(v) \equiv 0$, and we note that the case when $\underset{\sim}{b}(v) \not\equiv 0$ requires only a slight modification of our arguments below.

To show that (4.2) admits a solution we modify the diffusion matrix (4.4) by introducing an artificial viscosity term. Specifically, we consider the family of modified problems:

$$\begin{aligned} &B^{\varepsilon}(v^{\varepsilon}) \frac{dv^{\varepsilon}}{dt} = \nabla_v \bar{\Lambda}(v^{\varepsilon}) \\ &\lim_{t \to \pm\infty} v^{\varepsilon}(t) = v_{\pm}, \end{aligned} \tag{4.5}$$

where

$$B^{\varepsilon}(v) = \begin{bmatrix} \varepsilon a(v) I_p & 0 \\ 0 & b(v) \end{bmatrix}, \tag{4.6}$$

and where $a(v) > 0$ is to be determined. Below we show with reasonable hypothesis that the modified problem (4.5) has a solution for any $\varepsilon > 0$, and we show that the family $\{v^{\varepsilon}\}_{\varepsilon>0}$ satisfies uniform estimates that allows for the passage to the vanishing artificial viscosity limit.

Before giving the main results of this section we introduce some further notation. Partition a vector $v \in \mathbb{R}^n$ as $v = (v_1, v_2)^T$, where $v_1 \in \mathbb{R}^p$ and $v_2 \in \mathbb{R}^{n-p}$. For a smooth function $\Lambda : \mathbb{R}^n \to \mathbb{R}$, let Λ_{v_1} represent a

vector function with elements $\frac{\partial}{\partial v_i}\Lambda$, $1 \leq i \leq p$, and let Λ_{v_2} represent a vector function with elements $\frac{\partial}{\partial v_i}\Lambda$, $p+1 \leq i \leq n$. The notation $\Lambda_{v_1 v_1}$ shall represent a $(p \times p)$ matrix with elements $\frac{\partial^2}{\partial v_i \partial v_j}\Lambda$, $1 \leq i,j \leq p$, and $\Lambda_{v_2 v_1}$ shall represent an $(n-p) \times p$ matrix with elements $\frac{\partial^2}{\partial v_i \partial v_j}\Lambda$, $p+1 \leq i \leq n$, $1 \leq j \leq p$. Finally, for any square matrix $M(v) \in \mathbb{R}^{m\times m}$ we define the scalar-valued function $\mathcal{Q}_M(v)$ by

$$\mathcal{Q}_M(v) = \min_{\substack{\xi \in \mathbb{R}^m \\ |\xi|=1}} \xi^T M(v)\xi \ .$$

We are now ready to address the question of the existence of solutions to the modified problem (4.5). The particular list of assumptions given below are chosen for convenience only. In the next section we show they are easily applied to the compressible Navier-Stokes equations.

Lemma 4.1. As throughout, suppose that all of the basic preliminary assumptions of Section 2 are satisfied. In particular, assume that $u_- \in \mathcal{R}$ and $u_+ \in \Gamma_k^-(u_-)$, where $\Gamma_k^-(u_-)$ is a branch of the Hugoniot locus on which the entropy condition (2.5) is satisfied. Moreover, assume the following:

(a) $$\overline{G'} \subset \nabla E(\mathcal{R}) \ ,$$

where here $G' \subset \mathbb{R}^n$ is the set of states that can be reached by any smooth path $y(t)$ with $y(0) = v_-$, $\frac{d}{dt}\overline{\Lambda}(y(t)) > 0$ and $0 < \overline{\Lambda}(y(t)) < \Lambda(u_+)$. Second, assume that $\mathcal{Q}_b(v) > 0$ and $a(v)$ of the modified viscosity matrix (4.6) is constructed so as to satisfy

(b) $$c_1 \leq a(v)\mathcal{Q}_{b^{-1}}(v) \leq c_2 \ ,$$

for some positive constants c_1 and c_2 any every $v \in \nabla E(\mathcal{R})$. Finally, assume that for all $v \in \overline{G'}$ we have

(c) $$[\mathcal{Q}_b(v)\mathcal{Q}_{b^{-1}}(v)]^{1/2} \geq g'((\varepsilon|v_1|^2 + |v_2|^2)^{1/2}) \ ,$$

for some smooth nondecreasing function $g(r)$ with $\lim_{r\to+\infty} g(r) = +\infty$. Then for

any $\varepsilon > 0$, the modified travelling-wave equation (4.5) has a solution, and denoting this solution by $v^{\varepsilon}(t)$, we have the estimate:

$$g((\varepsilon|v_1^{\varepsilon}(t)|^2 + |v_2^{\varepsilon}(t)|^2)^{1/2}) \leq g(\max(\varepsilon,1)|v_-|) + \Lambda(u_+) .$$

Proof: Observe that for any $\xi \in \mathbb{R}^n$ we have that

$$\mu(u(v))\xi^T B^{\varepsilon}(v)\xi \geq |H(u_-,u_+;u)|^2[\frac{1}{\varepsilon} Q_{a^{-1}}(v)|\hat{n}_1|^2 + Q_{b^{-1}}(v)|\hat{n}_2|^2]$$

$$\times [Q_a(v)|\tilde{\xi}_1|^2 + Q_b(v)|\tilde{\xi}_2|^2] ,$$

where $|\hat{n}_1|^2 + |\hat{n}_2|^2 = 1$ and where we use the notation $\tilde{\xi} = (\sqrt{\varepsilon}\ \xi_1,\xi_2)^T$. Using condition (b) one easily shows that the right hand side above dominates

$$|H(u_-,u_+;u)|^2\min\left(\frac{1}{\varepsilon c_2},\ 1\right)\min(c_1,Q_b(v)Q_{b^{-1}}(v))|\tilde{\xi}|^2 ,$$

and condition (c) gives us that this dominates

$$c[|H(u_-,u_+;u)|g'(|\tilde{v}|)|\tilde{\xi}|]^2 ,$$

for some positive constant c. (Note that above we have assumed that $g'(|\tilde{v}|) \leq c_1$, *of course we lose nothing by doing this.*) Recalling that Lemma 2.1 gives us that $|H(u_-,u_+;u)| \geq c(u_-,u_+,\mathcal{O})$ for u outside any open set containing u_- and u_+, we set $\xi = \frac{dv^{\varepsilon}}{dt}$ and mimic the proof of Theorem 2. This completes the proof of the lemma.

The estimate of Lemma 4.1 establishes that $|v_2^{\varepsilon}(t)|$ is bounded independent of ε; (provided that condition (c) does not depend on ε.) With an additional hypothesis we can show that $|v_1^{\varepsilon}(t)|$ remains bounded independent of ε and therefore obtain a uniform maximum-norm estimate.

Lemma 4.2. In addition to the hypotheses of Lemma 4.1, assume the following:

Suppose there exists a compact set $\Omega \subset \mathbb{R}^n$ of the form:

$$\Omega = \{v : v_1 \in \Omega_1^R\} \cap \{v : |v_2| \leq R\} \cap \bar{\mathcal{G}}'$$

where R is taken large enough so that

$$g(R) > g(\max(\varepsilon,1)|v_-|) + \Lambda(u_+) ,$$

$\Omega_1^R \subset \mathbb{R}^p$ contains $(v_1)_-$ and $(v_1)_+$, and Ω_1^R is such that for any $v^* \in \overline{G}'$ with $v_1^* \in \partial\Omega_1^R$, $|v_2^*| < R$ we have that

$$\bar{\Lambda}_{v_1}(v^*) \cdot n_{\Omega_1^R}$$

is of one sign. ($n_{\Omega_1^R} \in \mathbb{R}^p$ is the outward unit normal to Ω_1^R.) Then $v^\varepsilon(t)$ remains trapped in Ω for all $\varepsilon > 0$.

Proof: By the estimate of the previous lemma we have that $g((\varepsilon|v_1^\varepsilon(t)|^2 + |v_2^\varepsilon(t)|^2)^{1/2}) < g(R)$ and this implies $|v_2^\varepsilon(t)| < R$. Using the differential equations (4.5) and the assumption of this lemma we have that Ω_1^R is either forward or backward invariant with respect to $v_1^\varepsilon(t)$. Given $\varepsilon > 0$ suppose that $v_1^\varepsilon(t) \not\in \Omega_1^R$ for some t. If this were the case then we could not have $\lim_{t\to\pm\infty} v^\varepsilon(t) = v_\pm$. But this violates the result of Lemma 4.1 and therefore establishes the result of the present lemma.

The next lemma implies the result of Lemma 4.2 as well as giving a uniform maximum-norm estimate for $\frac{dv^\varepsilon}{dt}$.

Lemma 4.3. In addition to the hypotheses of Lemma 4.1, suppose we could find a smooth function $h : \mathbb{R}^{n-p} \to \mathbb{R}^p$ such that the surface

$$\Sigma_1 = \{(h(v_2), v_2) : v_2 \in \mathbb{R}^{n-p}\} \cap \overline{G}' \quad ,$$

contains v_- and v_+ and is such that $\bar{\Lambda}_{v_1}(v) = 0$, (i.e. $H_1(u_-, u_+; u(v)) = 0$), for every $v \in \Sigma_1$. Furthermore, suppose that for every $v \in \Sigma_1$, the $(p \times p)$ matrix

$$[(-s(u_-,u_+)I + Df(u(v)))(D^2E(u(v)))^{-1}]_{i,j} \quad 1 \le i,\ j \le p \quad ,$$

$$(= \bar{\Lambda}_{v_1 v_1}(v))$$

is either positive or negative definite. Then $v^\varepsilon(t)$ satisfies the uniform estimates:

(a) $$|v^\varepsilon|_\infty \le K_1 \;,$$

(b) $$\left|\frac{dv^{\varepsilon}}{dt}\right|_{\infty} < K_2 ,$$

for some positive constants K_1 and K_2.

Proof: We first prove (a). The idea is to show that $v^{\varepsilon}(t)$ stays in a small neighborhood of Σ_1 for $0 < \varepsilon < \varepsilon_0$ with ε_0 sufficiently small. Set

$$\Sigma_1^{\delta} = \{v : d(v,\Sigma_1) < \delta\} ,$$

with δ small but fixed and define

$$\Sigma_2^{K\varepsilon} = \{v : |\bar{\Lambda}_{v_1}(v)| < a(v)K\varepsilon\} .$$

Defining R as in the previous lemma choose $K\varepsilon$ sufficiently small so that a connected branch of

$$\Sigma_2^{K\varepsilon} \cap \{v : |v_2| < R\} \cap \bar{G}'$$

is contained in Σ_1^{δ}. Next, let n represent the outward unit normal of $\Sigma_2^{K\varepsilon}$, and compute that along $\partial\Sigma_2^{K\varepsilon} \cap \{v : |v_2| < R\} \cap \bar{G}'$, n is given by

$$|\bar{\Lambda}_{vv}(\bar{\Lambda}_{v_1},0)^T|^{-1}\cdot(\bar{\Lambda}_{v_1v_1}\bar{\Lambda}_{v_1},\bar{\Lambda}_{v_2v_1}\bar{\Lambda}_{v_1})^T .$$

Dotting (4.5) with n gives

$$\frac{dv^{\varepsilon}}{dt}\cdot n = |\bar{\Lambda}_{vv}(\bar{\Lambda}_{v_1},0)^T|^{-1}|\bar{\Lambda}_{v_1}|$$

$$\times\ [\xi^T\bar{\Lambda}_{v_1v_1}\xi\,\frac{|\bar{\Lambda}_{v_1}|}{\varepsilon a} + (b^{-1}\bar{\Lambda}_{v_2})^T\bar{\Lambda}_{v_2v_1}\xi] ,$$

where $\xi = \bar{\Lambda}_{v_1}/|\bar{\Lambda}_{v_1}|$. Since along $\partial\Sigma_2^{K\varepsilon}$ we have $|\bar{\Lambda}_{v_1}| = aK\varepsilon$, and since the hypotheses of the lemma gives us that $\bar{\Lambda}_{v_1v_1}$ is either positive or negative definite near Σ_1, we have that the bracketed term above can be made to have one sign. This is accomplished for all $v \in \partial\Sigma_2^{K\varepsilon} \cap \{v : |v_2| < R\} \cap \bar{G}'$ by choosing K sufficiently large when ε is sufficiently small. Following the reasoning of the proof of Lemma 4.2 completes the proof of (a).

To prove (b) observe that

$$\left|\frac{d}{dt}\, v_1^{\varepsilon}(t)\right| = \frac{1}{\varepsilon a(v)}\, |\bar{\Lambda}_{v_1}(v)| .$$

Since $v^{\varepsilon}(t) \in \Sigma_2^{K\varepsilon} \cap \{v : |v_2| < R\} \cap \bar{G}'$ we have $|\frac{d}{dt} v_1^{\varepsilon}(t)| < K$ for all sufficiently small ε and all t. Combining this with the trivial inequality $|\frac{d}{dt} v_2^{\varepsilon}(t)| < |b^{-1}(v)\bar{\Lambda}_{v_2}(v)|$, completes the proof of (b).

We are now ready to state and prove the main result of this section.

Theorem 3. Suppose that the assumptions of Lemma 4.1 and Lemma 4.3 are satisfied. Then the partial differential equation (1.1) with singular viscosity has a smooth travelling-wave solution $u(x - st)$ which is the limit of a sequence of artificial viscosity approximations.

Proof: From Lemma 4.1 we have a family of smooth solutions to (4.5^{ε}) which we denote by $\{v^{\varepsilon}\}_{\varepsilon>0}$. Normalize these by translating t so that

$$\bar{\Lambda}(v^{\varepsilon}(0)) = \frac{1}{2} \Lambda(u_+) \ .$$

Moreover, from Lemma 4.3 and the Arzela-Ascoli Theorem we have a continuous function $v(t)$ and a subsequence $v^{\varepsilon_k}(t)$ such that

$$v(t) = \lim_{\varepsilon_k \to 0} v^{\varepsilon_k}(t) \ ,$$

the convergence being uniform on compact t intervals. From the proof of Lemma 4.3 we also have that

$$H_1(u_-,u_+;u(v(t))) = \lim_{\varepsilon_k \to 0} \bar{\Lambda}_{v_1}(v^{\varepsilon_k}(t)) = 0 \ ,$$

and the usual bootstrap arguments from the theory of ordinary differential equations gives us that $v_2(t)$ is a smooth solution of

$$b(v) \frac{dv_2}{dt} = H_2(u_-,u_+;u(v)) \ .$$

Since by assumption $\bar{\Lambda}_{v_1 v_1}$ is nonsingular for $v \in \Sigma_1$, we have from the implicit function theorem that $v_1(t)$ is smooth as well. Therefore, $v(t)$

is a smooth solution of

$$B^0(v)\,\frac{dv}{dt} = H(u_-,u_+;u(v)) .$$

What remains to be shown is that $\lim_{t\to\pm\infty} v(t) = v_\pm$. With this in mind compute that

$$\frac{d}{dt}\,\bar{\Lambda}(v(t)) = \bar{\Lambda}_{v_2}(v(t)) \cdot b^{-1}(v(t))\bar{\Lambda}_{v_2}(v(t)) . \tag{4.7}$$

Therefore, $\bar{\Lambda}(v(t))$ is a nondecreasing function, and because it is the limit of $\bar{\Lambda}(v^{\varepsilon_k}(t))$, which is bounded between 0 and $\Lambda(u_+)$, we find that

$$0 \leq \bar{\Lambda}(v(t)) \leq \Lambda(u_+) .$$

Bounded monotonic sequences have limits, therefore

$$\lim_{t\to\pm\infty} \Lambda(v(t)) = \Lambda_\pm ,$$

exist. Now define $v^{(n)}(t)$ by

$$v^{(n)}(t) = v(t + n) ,$$

and again appealing to the Arzela-Ascoli Theorem, we have a continuous function $v^*(t)$ and a subsequence $n_k \to \infty$, such that

$$v^*(t) = \lim_{n_k\to\infty} v^{(n_k)}(t) ,$$

which also gives us that

$$\Lambda_+ = \lim_{n_k\to\infty} \bar{\Lambda}(v(t + n_k)) = \bar{\Lambda}(v^*(t)) .$$

Integrating (4.7) from $t = n_k$ to $t = 1 + n_k$, we have after changing variables

$$\bar{\Lambda}(v^{(n_k)}(1)) - \bar{\Lambda}(v^{(n_k)}(0))$$

$$= \int_0^1 \bar{\Lambda}_{v_2}(v^{(n_k)}(t)) \cdot b^{-1}(v^{(n_k)}(t))\bar{\Lambda}_{v_2}(v^{(n_k)}(t))dt .$$

Letting $n_k \to \infty$ implies that

$$0 = \bar{\Lambda}_{v_2}(v^*(t)) \cdot b^{-1}(v^*(t))\bar{\Lambda}_{v_2}(v^*(t)) ,$$

and therefore

$$\bar{\Lambda}_{v_2}(v^*(t)) = H_2(u_-,u_+;u(v^*(t))) = 0 .$$

Combining this with the fact that $H_1(u_-,u_+;u(v^*(t))) = 0$, we have that $u(v^*(t))$ is one of the two critical points of $H(u_-,u_+;u)$. However because of the normalization, that is $\bar{\Lambda}(v(0)) > \frac{1}{2}\Lambda(u_+)$, we must have that $v^*(t) \equiv v_+$. The same argument can be applied equally well to any subsequence of $v^{(n)}(t) = v(t + n)$ which shows that $\lim_{t\to\infty} v(t) = v_+$. Similarly, $\lim_{t\to-\infty} v(t) = v_-$ and so the proof of the theorem is complete.

§5. APPLICATION TO THE NAVIER-STOKES EQUATIONS

In this section we apply Theorem 3 to the compressible Navier-Stokes equations. From our analysis we conclude that these equations admit a smooth travelling-wave which is the limit of certain artificial viscosity approximations. Throughout this section we shall assume that

$$u_- = \begin{bmatrix} \rho_- \\ m_- \\ e_- \end{bmatrix} \in \mathcal{R} = \{(\rho,m,e)^T : \rho > 0,\ P > 0\} ,$$

and we assume that $u_+ \in \Gamma_k^-(u_-)$ where $k = 1$ or $k = 3$ (the genuinely nonlinear fields); see equation (2.8). Moreover, because the Navier-Stokes equations are invariant under a Galilean change of coordinates, we lose no generality by taking the shock speed s equal to zero.

We begin by explicitly transforming the travelling-wave equations for Navier-Stokes into the coordinate system given by the gradient of its entropy

function

$$E(u) = -\rho \log\big(P/((\gamma-1)\rho^{\gamma})\big) .$$

Computing some derivatives we find that

$$v_1 = \frac{\partial E}{\partial \rho} = -\log(2\rho e - m^2) - 2\rho e/(2\rho e - m^2) + (\gamma + 1)(\log(\rho) + 1) + \log 2 ,$$

$$v_2 = \frac{\partial E}{\partial m} = 2\rho m/(2\rho e - m^2) \quad \left(= (\gamma - 1)\frac{m}{P}\right) ,$$

$$v_3 = \frac{\partial E}{\partial e} = -2\rho^2/(2\rho e - m^2) \quad \left(= -(\gamma - 1)\frac{\rho}{P}\right) .$$

Inverting these equations we obtain

$$\rho(v) = (-v_3)^{\frac{-1}{\gamma-1}}\exp\left[\frac{1}{\gamma - 1}(v_1 + v_2^2/(-2v_3) - \gamma)\right] ,$$

$$m = \rho(v)v_2/(-v_3) ,$$

$$e = \rho(v)(1 + v_2^2/(-2v_3))/(-v_3) .$$

Therefore

$$f_1 = m = \rho(v)v_2/(-v_3) ,$$

$$f_2 = \frac{m^2}{\rho} + P = \rho(v)(1 - \gamma + v_2^2/v_3)/v_3 ,$$

$$f_3 = \frac{m}{\rho}(e + P) = \rho(v)v_2\big(\gamma/v_3 - \tfrac{1}{2}(v_2/v_3)^2\big)/v_3 ,$$

and it is easily verified that

$$f(u) - f(u_-) = \nabla_v \bar{\Lambda}(v) ,$$

where

$$\bar{\Lambda}(v) = (\gamma - 1)\rho(v)v_2/(-v_3) - f(u_-) \cdot v + \text{const} ,$$

and where the constant is chosen so that $\bar{\Lambda}(v_-) = 0$. Moreover, a rather lengthy calculation will reveal that

$$B(v) \equiv A(u(v))(D^2E(u(v)))^{-1}$$

$$= \mu \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1/v_3 & v_2/v_3^2 \\ 0 & v_2/v_3^2 & -v_2/v_3^3 + \frac{\lambda}{c_v} 1/v_3^2 \end{bmatrix} .$$

(The viscosity matrix $A(u)$ for Navier-Stokes is given in Section 1.) Note that in the v-coordinate system the region of physically admissible states is $\nabla E(R) = \{v : v_3 < 0\}$.

The result of this section is obtained by establishing the hypotheses of Theorem 3. To this end we first establish hypothesis (a) of Lemma 4.1. That is we show that the set G' is contained in a closed subset of $\nabla E(R)$. (Recall that G' is the set of all states that can be reached by any smooth path $y(t)$ with $y(0) = v_-$, $\frac{d}{dt}\bar{\Lambda}(y(t)) > 0$ and which satisfies $0 < \bar{\Lambda}(y(t)) < \bar{\Lambda}(v_+)$.) We accomplish this by *constructing* what we call a "Λ-wall". Specifically, we show that when $k = 1$, $m_- > 0$ (The case $k = 3$, $m_- < 0$ being similar) there exists a closed set of the form

$$W = \{v : v_2 > \alpha > 0,\ v_3 < \beta < 0\} ,$$

with $v_- \in W^0$, $v_+ \in \partial W$ such that for any $v^* \in \partial W \backslash \{v_+\}$ we have $\bar{\Lambda}(v^*) > \bar{\Lambda}(v_+)$. Having this wall W implies that $\bar{G}' \subset \nabla E(R)$.

Remark 5.1. The entropy condition (2.5) implies that the momentum component of u_- (i.e. m_-) for the Navier-Stokes 1 or 3 zero-speed-wave can never be zero. See (2.7) and compare it with the entropy condition (2.5).

To see that W can be made to have the properties described above, set

$$\alpha = (v_2)_+ ,$$

and

$$\beta = -4(\gamma - 1)/(f_2(u_-)/m_-)^2 .$$

(From the remark above $(f_2(u_-)/m_-)^2$ is positive and bounded.) By checking the sign of $\frac{\partial}{\partial v_3}\bar{\Lambda}(v^*)$ for

$$v^* \in \left\{v : \frac{\partial}{\partial v_1}\bar{\Lambda}(v) = 0, \quad v_2 = (v_2)_+, \quad v_3 < 0\right\},$$

it is clear that $\bar{\Lambda}(v^*) > \Lambda(v_+)$ when $v^* \neq v_+$; see Figure 1. Moreover, it is easy to check that $\frac{\partial}{\partial v_2}\bar{\Lambda}(v^*) > 0$ for

$$v^* \in \left\{v : \frac{\partial}{\partial v_1}\bar{\Lambda}(v) = 0,\ v_2 > (v_2)_+\ ,\ v_3 = -\frac{4(\gamma - 1)}{(f_2(u_-)/m_-)^2}\right\},$$

so we have $\Lambda(v^*) > \Lambda(v_+)$ here as well; again see Figure 1. Finally, $\frac{\partial}{\partial v_1}\bar{\Lambda}(v^*)$ is positive (resp. negative) if $v^* \in \partial W$ with v^* lying above (resp. below) the surface $\left\{v : \frac{\partial}{\partial v_1}\bar{\Lambda}(v) = 0,\ v_2 > 0,\ v_3 < 0\right\}$, so we conclude that $\bar{\Lambda}(v^*) > \Lambda(v_+)$ for all $v^* \in \partial W\backslash\{v_+\}$.

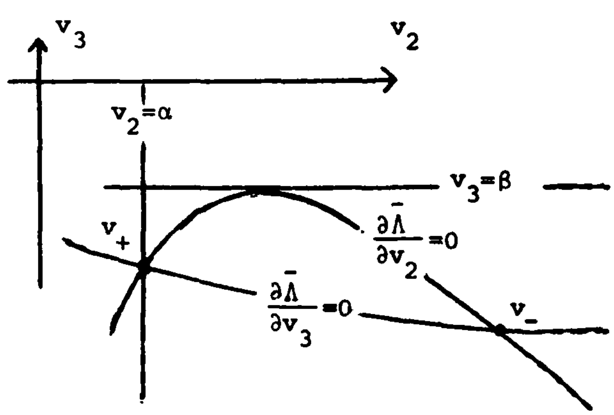


Figure 1

Next we establish hypotheses (b) and (c) of Lemma 4.1. Define the viscosity submatrix

$$b(v) = \mu \begin{bmatrix} -1/v_3 & v_2/v_3^2 \\ v_2/v_3^2 & -v_2/v_3^3 + \frac{\lambda}{c_v} 1/v_3^2 \end{bmatrix},$$

and construct the artificial viscosity $\varepsilon a(v)$ $(= B^{\varepsilon}_{1,1}(v))$ so that for all $v \in \{v : v_3 < 0\}$

$$c_1 < a(v) Q_{b^{-1}}(v) < c_2 ,$$

where c_1 and c_2 are some positive constants. Note that for a symmetric matrix S, Q_S is given by the smallest eigenvalue of S.

Remark 5.2. This particular artificial viscosity has the form (1.5) when written in conserved variables.

An elementary calculation will show that

$$Q_b(v) Q_{b^{-1}}(v) > \frac{\left(\frac{v_2^2}{|v_3|} + \frac{\lambda}{c_v}\right)|v_3| + v_2^2}{\left(\frac{v_2^2}{|v_3|} + \frac{\lambda}{c_v} + |v_3|\right)^2},$$

and for any $v \in W$ this dominates

$$\frac{c^2}{(1 + |v_2| + |v_3|)^2},$$

for some positive constant c. With this estimate we can satisfy hypothesis (c) of Lemma 4.1 by choosing $g(r) = c \log(1 + r)$.

To conclude this section we show that the Navier-Stokes equations satisfy the assumptions of Lemma 4.3. Solving the equation $\frac{\partial}{\partial v_1} \bar{\Lambda}(v) = 0$ for v_1 we find that

$$v_1 = h(v_2,v_3) = -\frac{v_2^2}{(-2v_3)} + \gamma + (\gamma - 1)\log\left(\frac{(-v_3)^{\frac{\gamma}{\gamma-1}}}{v_2} m_-\right),$$

and clearly this is smooth in $\mathcal{W}$. Finally for $v = (h(v_2,v_3),v_2,v_3)$, that is for v such that $\frac{\partial}{\partial v_1}\bar{\Lambda}(v) = 0$, we have that

$$\frac{\partial^2}{\partial v_1^2}\bar{\Lambda}(v) = \frac{m_-}{(\gamma - 1)},$$

and as Remark 5.1 points out, m_- can never be zero for a zero speed 1 or 3 wave. Therefore we have established all of the hypotheses of Theorem 3 and hence conclude the result of Theorem 3 for the Navier-Stokes equations.

References

1. C. Bardos and F. Golse, On some features of the notion of entropy at the level of Boltzman and Navier-Stokes equation. C. R. Acad. Sc. Paris, t. 299, Série I, no 7, 1984, 225-228.

2. C. Conley and J. Smoller, Shock waves as progressive wave solutions of higher-order equations. Comm. Pure Appl. Math. 24 (1971), 459-472.

3. C. Conley and J. Smoller, Shock waves as progressive wave solutions of higher-order equations, II. Comm. Pure Appl. Math. 25 (1972), 131-146.

4. R. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves. Wiley-Interscience: New York 1948.

5. L. R. Foy, Steady state solutions of hyperbolic systems of conservation laws with viscosity terms. Comm. Pure Appl. Math. 17 (1964), 177-188.

6. D. Gilbarg, The existence and limit behavior of one-dimensional shock layers. Amer. J. Math. 73 (1951), 256-274.

7. V. Guillemin and A. Pollack, Differential Topology. Prentice-Hall: Englewood Cliffs, NJ (1974).

8. P. D. Lax, Hyperbolic systems of conservation laws, II. Comm. Pure Appl. Math. 10 (1957), 532-566.

9. P. D. Lax, Shock waves and entropy, in "Proceedings, Symposium at University of Wisconsin, 1971" (E. H. Zarantonello, Ed.), 603-634.

10. M. S. Mock, Discrete shocks and genuine nonlinearity, Mich. Math. J. 25 (1978), 131-146.

11. M. S. Mock, A topological degree for orbits connecting critical points of autonomous systems, J. Diff. Eq. 38 (1980), 176-191.

12. J. Smoller, Shock Waves and Reaction-Diffusion Equations. Springer-Verlag: New York 1983.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 2906	2. GOVT ACCESSION NO. AD-A167518	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ON TRAVELLING-WAVE SOLUTIONS TO SYSTEMS OF CONSERVATION LAWS WITH SINGULAR VISCOSITY		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Howard Prue and Richard Sanders		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041 MCS 82-00676
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53705		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below.		12. REPORT DATE January 1986 13. NUMBER OF PAGES 31
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

compressible Navier-Stokes equations
entropy inequality
artificial viscosity
modulo-2 intersection number

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The method of vanishing artificial viscosity is used to obtain smooth, large-data travelling-wave solutions to a class of conservation laws with semidefinite viscosity. The one-dimensional Navier-Stokes equations serve as an illustrating example.

END
FILMED

6-86

DTIC